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SIMILAR CONICS THROUGH THREE POINTS*

BY

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In a recent number of the Transactions (vol. 4, 1903, p. 103), Professor R. E. ALLARDICE has determined the envelope of the principal axes of a set of similar conics through three points. It may be of interest to point out that the final equation found by him (p. 105, near the foot) can be put in a slightly different form (see equation (8) below), which has the advantage of showing at a glance that the line infinity is a double tangent at the circular points; it is easy to pass directly from that equation to the form here proposed, by algebraic manipulations, but an alternative investigation, *ab initio*, will be given.

Take the center of the circumcircle of the three given points as origin, and let its radius be taken as unit; then if x, y are conjugate complex coördinates, the circumcircle is

$$(1) \qquad xy = 1, \qquad \text{or} \qquad |x| = 1.$$

Let the three given points be $x = a, b, c$; and let any particular conic of the set be

$$(2) \qquad (x - ty - p)^2 + \lambda(x + ty - q)^2 = r$$

the axes being $x - ty - p = 0$, $x + ty - q = 0$, where $|t| = 1$. Then the condition of similarity is simply that λ is constant; in ALLARDICE'S notation,† $(s - 1)/(s + 1)$ is equal to λ or $1/\lambda$.

The conic (2) cuts the circle (1) in four points, of which three are $x = a, b, c$; let the fourth be $x = d$. Then, if, following MORLEY,‡ we write

$$(3) \qquad s_1 = a + b + c, \qquad s_2 = bc + ca + ab, \qquad s_3 = abc$$

it will be found that a, b, c, d are the roots of

$$(4) \qquad (x^2 - px - t)^2 + \lambda(x^2 - qx + t)^2 - rx^2 = 0$$

and so

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† Hereafter to be indicated by A.

‡ Compare, e. g., these Transactions, vol. 1 (1900), p. 100; vol. 4 (1903), p. 1.

$$(5)\quad \begin{cases} (1+\lambda)(s_1+d)=2(p+\lambda q), \\ (1+\lambda)(s_3+s_2d)=2t(-p+\lambda q), \\ s_3d=t^2. \end{cases}$$

Hence we find

$$(6)\begin{cases} 4pt=(1+\lambda)(s_1t+t^3/s_3-s_3-s_2t^2/s_3)=(1+\lambda)(t-bc)(t-ca)(t-ab)/s_3, \\ 4\lambda qt=(1+\lambda)(s_1t+t^3/s_3+s_3+s_2t^2/s_3)=(1+\lambda)(t+bc)(t+ca)(t+ab)/s_3, \end{cases}$$

and so the two axes are

$$(7)\quad \begin{cases} x-ty=\tfrac{1}{4}(1+\lambda)(s_1+t^2/s_3-s_3/t-s_2t/s_3) \\ \qquad =\tfrac{1}{4}(1+\lambda)(t-bc)(t-ca)(t-ab)/s_3t, \\ x+ty=\tfrac{1}{4}(1+1/\lambda)(s_1+t^2/s_3+s_3/t+s_2t/s_3) \\ \qquad =\tfrac{1}{4}(1+1/\lambda)(t+bc)(t+ca)(t+ab)/s_3t. \end{cases}$$

From equations (7) it is clear that the envelope of one axis is found from that of the other by changing λ to $1/\lambda$ (or, in A., by changing the sign of s).

Taking the first of equations (7), we see at once that the envelope is a deltoid (3-cusp hypocycloid) with its center at the point $x=\frac{1}{4}(1+\lambda)s_1$; that the cuspidal tangents are given by $t^3=s_3^2$, and the tangents at the vertices by $t^3=-s_3^2$; and that the radius of the vertex-circle is $\frac{1}{4}|1+\lambda|$, while the radius of the cusp-circle is $\frac{3}{4}|1+\lambda|$. Thus the deltoid is completely specified.

The three tangents from the circumcenter $(x=0, y=0)$ are given by $t=bc$, ca, ab, i. e., the tangents are $x-bcy=0$, etc.; and so these three tangents are perpendicular to the sides of the triangle (A., p. 106).

To transform the first of equations (7) to ALLARDICE's form, we observe that if X, Y, Z are trilinears, then

$$X=\tfrac{1}{2}\,[x+bcy-(b+c)]/\sqrt{bc}, \text{ etc.}$$

and so $uX+vY+wZ=0$ gives

$$\left(\frac{u}{\sqrt{bc}}+\frac{v}{\sqrt{ca}}+\frac{w}{\sqrt{ab}}\right)x+(u\sqrt{bc}+v\sqrt{ca}+w\sqrt{ab})y$$

$$-\left[u\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+v\left(\sqrt{\frac{c}{a}}+\sqrt{\frac{a}{c}}\right)+w\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)\right]=0,$$

or say

$$Px+Qy-R=0.$$

If we compare this with the first equation of (7), we find the line-equation of the envelope

$$PQR = \tfrac{1}{4}(1 + \lambda)(Q + Pbc)(Q + Pca)(Q + Pab)/abc.$$

Now*

$$Q + Pbc = 2\sqrt{bc}(u - v\cos C - w\cos B) \text{ etc.}$$

$$PQ = u^2 + v^2 + w^2 - 2vw\cos A - 2wu\cos B - 2uv\cos C$$

$$R = -2(u\cos A + v\cos B + w\cos C),$$

thus the trilinear line-equation is

$$(8)\ 0 =$$

$$(u\cos A + v\cos B + w\cos C)(u^2 + v^2 + w^2 - 2vw\cos A - 2wu\cos B - 2uv\cos C)$$

$$+ (1+\lambda)(u - v\cos C - w\cos B)(v - w\cos A - u\cos C)(w - u\cos B - v\cos A).$$

Since $\lambda = (s-1)/(s+1)$, we have $\lambda + 1 = 2s/(s+1)$. Now compare (8) with the final equation (A., p. 105), and we see that the first term of (8) must be equal to

$$\Sigma[u(u - v\cos C - w\cos B)\{u\cos(B - C) - v\cos B - w\cos C\}]$$

$$-2(u - v\cos C - w\cos B)(v - w\cos A - u\cos C)(w - u\cos B - v\cos A)$$

an identity which is easily verified, although it can hardly be regarded as obvious *a priori*.

ALLARDICE remarks that in the special case $s = -1$, ($\lambda = 0$ or ∞) the deltoid is degenerate; but it may be of interest to add that in this case the conics of the set are all *parabolas;* and it is the envelope of the *infinite* axis which becomes degenerate; the envelope of the axes (in the ordinary sense) of the parabolas is not degenerate, and has its center at the point $x = \frac{1}{4}s_1$; the radius of its vertex-circle is $\frac{1}{4}$ and that of its cusp-circle is $\frac{3}{4}$; its cuspidal tangents are found as in the general case.

From equation (8) we see that the case $\lambda = -1$ is also degenerate; here the conics are all *circles*, and there is no proper envelope. However, equation (8) gives three points, the circumcenter and the two circular points. In this case (as well as when $s = \infty$, $\lambda = +1$, noticed by ALLARDICE) the two axes give the *same* envelope because $\lambda = 1/\lambda$.

QUEEN'S COLLEGE, GALWAY,
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* It will be found necessary to take $(b+c)/\sqrt{bc}$ equal to $-2\cos A$ (not $+2\cos A$), etc., in order to satisfy the conditions $A + B + C = \pi$, $\sqrt{bc}\cdot\sqrt{ca}\cdot\sqrt{ab} = +abc$.

ADDITION (*August* 12, 1903). If the conics are parabolas the following construction gives the vertex-circle very neatly: Bisect the sides of the given triangle in L, M, N; bisect the sides of LMN in P, Q, R; then the circumcircle of PQR is the vertex-circle. For L is given by $x = \frac{1}{2}(b + c)$, M by $x = \frac{1}{2}(c + a)$; so R is given by $x = \frac{1}{4}(c + s_1)$, which is on the vertex-circle $|x - \frac{1}{4}s_1| = \frac{1}{4}$.

This construction is due to Professor H. C. MCWEENEY (University College, Dublin); it can be extended to the general case, but the method is not so simple.